Galois cohomology seminar
Week 11 - Brauer group of a local field

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Contents

1 Motivation - application to Tate's theorem

First, let me remind you of the statement of Tate's theorem.

Theorem 1.1 (Tate's theorem)**.** *Let G be a finite group and A a G-module. Suppose that for all subgroups $H \subset G$ that*

1. $H^1(H, A) = 0$.
2. $H^2(H, A)$ *is cyclic of order* $|H|$.

Then a choice of generator γ of $H^2(G, A)$ induces an isomorphisms

$$\widehat{H}^i(G, \mathbb{Z}) \to \widehat{H}^{i+2}(G, A)$$

for all $i \in \mathbb{Z}$.

To construct the local Artin map in local class field theory, we will want to apply this in the following situation: K is a nonarchimedean, complete local field (such as $\mathbb{Q}_p$), L/K a finite Galois extension, $G = \operatorname{Gal}(L/K)$, and $A = L^\times$. By Galois theory, if $H \subset G$ is a subgroup, then $H = \operatorname{Gal}(L/E)$ for an intermediate field E, and then by Hilbert 90,

$$H^1(H, A) = H^1(\operatorname{Gal}(L/E), L^\times) = 0$$

The remaining part of the hypotheses of Tate's theorem that we need is

$$H^2(H, A) = H^2(\mathrm{Gal}(L/E), L^\times) \cong \mathbb{Z}/m\mathbb{Z}$$

where $m = |H| = [L : E]$. In principle, we need this for all subgroups H, so for all intermediate fields E. However, E is just another local field (with extended valuation from K), so it suffices to show

$$\mathrm{Br}(L/K) \cong H^2(\mathrm{Gal}(L/K), L^\times) \cong \mathbb{Z}/m\mathbb{Z}$$

The resulting isomorphism of Tate's theorem is called the local Artin map.

I include the reminder about the identification of this with the relative Brauer group just for flavor. We're going to prove this by purely group cohomological considerations, without thinking about central simple algebras. Along the way, we'll prove

$$\mathrm{Br}(K^{\mathrm{un}}/K) \cong \mathbb{Q}/\mathbb{Z}$$

From Igor's notes, we actually know that

$$\mathrm{Br}(K^{\mathrm{un}}/K) = \mathrm{Br}(K)$$

so one way to think of my talk to day is just as the computation of the Brauer group of a local field. You can do this from the algebras side, as done in Igor's notes [3], but I'm going to follow chapter 3 of Milne [2] instead.

Although eventually we want this to work for any finite Galois extension L/K, today we are just going to consider the case of an unramified extension L/K.

2 Brauer group of a local field

For the rest of the talk, let K be a complete nonarchimedean local field. If I just say "local field," I also mean nonarchimedean and complete.

2.1 Review of local fields - extending valuations, ramification, unramified extensions

Theorem 2.1. *Let K be a local field, and L/K a finite extension of degree n. Then the absolute value (or valuation) extends uniquely to L, and L is complete with respect to the extended value. In particular, the extended absolute value is described for $\beta \in L$ by*

$$|\beta|_L = |\,\mathrm{N}_K^L(\beta)|_K^{1/n}$$

Proof. Milne, *Algebraic Number Theory*, Theorem 7.38 [1]. □

Definition 2.2. Let K be a local field with nonarchimedean absolute value $|\cdot|_K$ and discrete valuation v_K. The **associated local ring** of K is

$$\mathcal{O}_K = \{x \in K : |x|_K \leq 1\} = \{x \in K : v_K(x) \geq 0\}$$

Since $\mathcal{O}_K$ has a discrete valuation, it is a local PID. The maximal ideal is

$$\mathfrak{m}_K = \{x \in K : |x|_K < 1\} = \{x \in K : v_K(x) > 0\}$$

The **residue field** of K is the quotient $k_K = \mathcal{O}_K/\mathfrak{m}_K$. A **uniformizer** π for K is a generator for $\mathfrak{m}_K$.

$$(\pi) = \mathfrak{m}_K$$

Definition 2.3. Let K be a complete nonarchimedean discretely valued field, and L/K a finite extension. Let k_K be the associated residue field of K and k_L the associated residue field of L. Note that $\mathcal{O}_K \subset \mathcal{O}_L$ and $\mathfrak{m}_K \subset \mathfrak{m}_L$, hence

$$k_K \hookrightarrow k_L$$

The **residual degree** is

$$f(L|K) = f_K^L = [k_K : k_L]$$

Definition 2.4. Let K be a complete nonarchimedean discretely valued field, and L/K a finite extension with $d = [L : K]$. Let $v_K : K^\times \to \mathbb{Z}$ be a normalized discrete valuation. Let $v_L : L^\times \to \mathbb{R}$ be the extension of v_K, and the we know that

$$\operatorname{im} v_L \subset \frac{1}{d}\mathbb{Z}$$

so v_L is also discrete. The **ramification degree** is

$$e(L|K) = e_K^L = e_{L/K} = [v_L(L^\times) : v_K(K^\times)]$$

That is, if π_K is a uniformizer for K and π_L is a uniformizer for L, then

$$(\pi_K) = \left(\pi_L^{e(L|K)}\right)$$

as ideals of $\mathcal{O}_L$.

Definition 2.5. Let $L, K, e_{L/K}, f_{L/K}$ be as above. If $e_{L/K} = 1$, then L/K is **unramified**. if $f_{L/K} = 1$, then L/K is **totally ramified**.

Remark 2.6. Today we're going to be dealing with primarily unramified extensions. This just means that a uniformizer π for K is also a uniformizer for L.

Proposition 2.7. *Let K be a local field, and L/K a finite extension. Then $[L : K] = ef$.*

Proposition 2.8. *Let K be a local field. There is a bijection*

$$\begin{array}{ccc} \{\text{finite unramified extensions of } K\} & \to & \{\text{finite extensions of } k_K\} \\ L = K(\alpha) & \mapsto & k_L = k_K(\overline{\alpha}) \end{array}$$

This correspondence preserves Galois groups. That is, Galois extensions L/K correspond to Galois extensions k_L/k_K, and corresponding Galois extensions have isomorphic Galois groups.

$$\operatorname{Gal}(L/K) \cong \operatorname{Gal}(k_L/k_K)$$

Remark 2.9. In our case, a local field K has a finite residue field k, and k has a unique finite extension of degree n, which is cyclic Galois, so K has a unique unramified extension K_n of degree n, which is cyclic Galois.

2.2 Cohomology computations

Definition 2.10. Let K be a local field with normalized discrete valuation $v : K^\times \to \mathbb{Z}$. Let $\mathcal{O}_K^\times$ be the group of units of $\mathcal{O}_K$.

$$\mathcal{O}_K^\times = \left\{x \in K^\times : v(x) = 0\right\} = \left\{x \in K^\times : |x| = 1\right\}$$

Fix a uniformizer π for K, so that $\mathfrak{m}_K = (\pi)$. Now set

$$\begin{aligned} U_K &= \mathcal{O}_K^\times \\ U_K^m &= 1 + \pi^m \mathcal{O}_K = \{1 + a\pi^m : a \in \mathcal{O}_K\} \qquad m \in \mathbb{Z}_{\geq 1} \end{aligned}$$

Notation: In all of the following lemmas, we fix notation as follows. K will be a local field, and L/K a finite unramified extension of degree $n = [L : K]$. The respective residue fields will be k, ℓ. We fix a uniformizer π for K (and L). Set $G = \mathrm{Gal}(L/K) \cong \mathrm{Gal}(\ell/k)$, and note that $G \cong \mathbb{Z}/n\mathbb{Z}$.

Lemma 2.11 (Milne [2], Chapter 3, Lemma 1.3)**.** *Let K, L, k, ℓ, π, G be as above. There are exact sequences of G-modules*

$$1 \longrightarrow U_L^1 \hookrightarrow U_L \xrightarrow{u \mapsto u \bmod \pi} \ell^\times \longrightarrow 1$$

$$1 \longrightarrow U_L^{m+1} \hookrightarrow U_L^m \xrightarrow{1 + a\pi^m \mapsto a \bmod \pi} \ell \longrightarrow 0$$

[1] *which induce isomorphisms (for all $m \in \mathbb{Z}_{\geq 1}$)*

$$U_L/U_L^1 \cong \ell^\times \qquad U_L^m/U_L^{m+1} \cong \ell$$

Proof. Exactness is easily checked by inspection. The isomorphisms come from the first isomorphism theorem. □

Remark 2.12. Recall that for a cyclic group (such as G), the Tate cohomology of any G-module A is 2-periodic,

$$\begin{aligned} \cdots \cong \widehat{H}^0(G, A) \cong \widehat{H}^2(G, A) \cong \cdots \\ \cdots \cong \widehat{H}^1(G, A) \cong \widehat{H}^3(G, A) \cong \cdots \end{aligned}$$

Recall that the **Herbrand quotient** of A is defined to be

$$h(A) = \frac{|\widehat{H}^0(G, A)|}{|\widehat{H}^1(G, A)|}$$

when things are finite. Recall a result that Stan proved, that if A is finite, then $h(A) = 1$. This implies that all the Tate cohomology groups of A have the same order.

$$\cdots = |\widehat{H}^0(G, A)| = |\widehat{H}^1(G, A)| = \cdots$$

[1] It may look funny to write 0 at one end of the sequence and 1 at the other, I've just done this to emphasize that $U_L^m, U_L, \ell^\times$ are written multiplicatively, while ℓ is written additively. Both 1 and 0 refer to the trivial group.

Lemma 2.13 (Milne [2], Chapter 3, Lemma 1.4)**.** *Let K, L, k, ℓ, G be as above.*

1. $\widehat{H}^i(G, \ell^\times) = 0$ *for all* $i \in \mathbb{Z}$*.*

2. *The norm map* $\mathrm{N}_k^\ell : \ell^\times \to k^\times$ *is surjective.*

Proof. (1) By Hilbert 90, $H^1(G, \ell^\times) = 0$. Since $\ell^\times$ is finite, by the previous remark/discussion, $h(\ell^\times) = 1$ so all the groups $\widehat{H}^i(G, \ell^\times)$ are zero.
(2) Recall that $\widehat{H}^0(G, A)$ is, by definition, $(\ell^\times)^G/N_G(\ell^\times) = k^\times/N_G(\ell^\times)$. Since field norm map agrees with the group norm map, and $\widehat{H}^0(G, \ell^\times) = 0$, this says that the norm map from $\ell^\times$ to $k^\times$ is surjective. □

Lemma 2.14 (Milne [2], Chapter 3, Lemma 1.5)**.** *Let K, L, k, ℓ, π, G be as above.*

1. $\widehat{H}^i(G, \ell) = 0$ *for all* $i \in \mathbb{Z}$*.*

2. *The trace map* $\mathrm{Tr} : \ell \to k$ *is surjective.*

Proof. (1) By the additive version of Hilbert 90, $H^i(G, \ell) = 0$ for $i > 0$. By periodicity (since G is cyclic), they then vanish for all i.
(2) As in the previous lemma, we consider $\widehat{H}^0(G, \ell) = \ell^G/N_G\ell = k/N_G\ell$. For additive structure of ℓ, the group norm map is the trace map. Since $\widehat{H}^0$ vanishes, the trace map is surjective. □

Proposition 2.15 (Milne [2], Chapter 3, Proposition 1.2)**.** *Let K, L, k, ℓ, G be as above.*

1. *The norm map* $\mathrm{N}_K^L : U_L = \mathcal{O}_L^\times \to U_K = \mathcal{O}_K^\times$ *is surjective.*

2. $\widehat{H}^0(G, U_L) = 0$*.*

Proof. (1) We combine our exact sequences from Lemma 2.11 into some commutative diagrams of G-modules. The maps on the right are surjective by the previous two lemmas.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_L^1 & \hookrightarrow & U_L & \longrightarrow & \ell^\times & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} & & \twoheaddownarrow{\scriptstyle \mathrm{N}} & & \\
1 & \longrightarrow & U_K^1 & \hookrightarrow & U_K & \longrightarrow & k^\times & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_L^{m+1} & \hookrightarrow & U_L^m & \longrightarrow & \ell & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{N}} & & \downarrow{\scriptstyle \mathrm{N}} & & \twoheaddownarrow{\scriptstyle \mathrm{Tr}} & & \\
1 & \longrightarrow & U_K^{m+1} & \hookrightarrow & U_K^m & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

We want to how $N : U_L \to U_K$ is surjective, so let $u \in U_K = \mathcal{O}_K^\times$, and let $\overline{u}$ be the image in $k^\times = (\mathcal{O}_K/\mathfrak{m}_K)^\times$. By surjectivity of $\mathrm{N} : \ell^\times \to k^\times$ and $U_L \to \ell^\times$, there exists $v_0 \in U_L = \mathcal{O}_L^\times$ such that $\mathrm{N}(\overline{v}_0) = \overline{u}$. That is, $u\,\mathrm{N}(v_0)^{-1} \in \ker(U_K \to k^\times)$, so by exactness, it is in the image, which is U_K^1.

Now we perform the same diagram chase on the second diagram. By surjectivity of $U_L^1 \to \ell \to k$, there exists $v_1 \in U_L^1$ such that $\mathrm{N}(\overline{v}_1) = \overline{u\,\mathrm{N}(v_0)^{-1}}$. Thus

$$u\,\mathrm{N}(v_0)^{-1}\,\mathrm{N}(v_1)^{-1} = u\,\mathrm{N}(v_0 v_1)^{-1} \in \ker(U_K^m \to k) = U_K^2$$

Repeating this inductively with the second diagram, we obtain $v_i \in U_L^i$ such that

$$u\,\mathrm{N}(v_0 v_1 \cdots v_i)^{-1} \in U_K^{i+1}$$

Now we claim that the sequence $a_n = \prod_{i=1}^{n} v_i$ converges (in U_L^1). Then because

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |v_n| = 1$$

Since U_L is complete, the Cauchy sequence a_n has a limit, so we can set

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \prod_{i=1}^{n} v_i$$

Then

$$u\,\mathrm{N}(a_n)^{-1} \in U_K^n = \bigcap_{i=1}^{n} U_K^i \implies u\,\mathrm{N}(a)^{-1} \in \bigcap_{i=1}^{\infty} U_K^i = \{1\}$$

so $u = \mathrm{N}(a)$. This proves the surjectivity that we wanted.
(2) By definition,

$$\widehat{H}^0(G, U_L) = U_L^{\mathrm{Gal}(L/K)}/N_G U_L = U_K/N_G U_L$$

Since the group norm coincides with field norm and by (1) the field norm is surjective here, $\widehat{H}^0(G, U_L) = 0$. □

Proposition 2.16 (Milne [2], Chapter 3, Proposition 1.1)**.** *Let K, L, π, G be as above. Then $\widehat{H}^i(G, U_L) = 0$ for all $i \in \mathbb{Z}$.*

Proof. Every element $\alpha \in L^\times$ can be written uniquely as $\alpha = u\pi^m$ where $u \in U_L = \mathcal{O}_L^\times, m = v_L(\alpha) \in \mathbb{Z}$, which gives an isomorphism

$$L^\times \to U_L \times \mathbb{Z} \qquad u\pi^m \mapsto (u, m)$$

Since L/K is unramified, $\pi \in K$ as well, so the $G = \mathrm{Gal}(L/K)$-action on $L^\times$ satisfies, for $\tau \in G$,

$$\tau(u\pi^m) = (\tau u)\pi^m$$

which is to say, the isomorphism $L^\times U_L \times \mathbb{Z}$ is an isomorphism of G-modules, with $\mathbb{Z}$ viewed as trivial G-module. Thus

$$\widehat{H}^i(G, L^\times) \cong \widehat{H}^i(G, U_L \times \mathbb{Z}) \cong \widehat{H}^i(G, U_L) \times \widehat{H}^i(G, \mathbb{Z})$$

For $i = 1$, the first of the above is zero by Hilbert 90. By Proposition 2.15, $\widehat{H}^0(G, U_L) = 0$. Then by 2-periodicity for G cyclic, we get $\widehat{H}^i(G, U_L) = 0$ for all $i \in \mathbb{Z}$. □

Corollary 2.17 (Milne [2], Chapter 3, Corollary 1.6)**.** *Let F/K be an infinite unramified Galois extension, and $G = \mathrm{Gal}(F/K)$. Then $H^i(G, U_F) = 0$ for $i \in \mathbb{Z}_{>0}$.*

Proof. This follows immediately from the previous proposition and the fact that cohomology for F is a direct limit of cohomology groups over finite unramified Galois extensions L/K.

$$H^i(\mathrm{Gal}(F/K), U_M) = \varinjlim H^i(\mathrm{Gal}(L/K), U_L) = \varinjlim 0 = 0$$

□

2.3 Relative Brauer group of maximal unramified extension, and invariant maps

Definition 2.18. Let L/K be an unramified (possibly infinite) extension, with v_L the extended valuation on L and Galois group $G = \mathrm{Gal}(L/K)$. From the short exact sequence

$$0 \to U_L \to L^\times \xrightarrow{v_L} \mathbb{Z} \to 0$$

(where $\mathbb{Z}$ is a tivial G-module) we get a long exact sequence on $H^i(G, -)$, where the $H^i(G, U_F)$ terms vanish, so we get an isomorphism

$$H^2(G, L^\times) \xrightarrow[\cong]{v_L} H^2(G, \mathbb{Z})$$

We can also consider the short exact sequence of trivial G-modules

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

In the associated long exact sequence, the $H^i(G, \mathbb{Q})$ terms vanish, because they are torsion (a direct limit of torsion groups is torsion) and they are "uniquely divisible," which is to say, they are $\mathbb{Q}$-vector spaces. A torsion group with a $\mathbb{Q}$-action must be zero. So the connecting homomorphism is an isomorphism.

$$H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\cong]{\delta} H^2(G, \mathbb{Z})$$

Recall that for trivial modules, H^1 is the same as Hom. Well actually, for profinite cohomology, it's continuous homomorphisms, so

$$H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}/\mathbb{Z})$$

Recall that G has a special element $\sigma = \mathrm{Frob}_{F/K}$, which restricts to the Frobenius automorphism on any finite unramified subextension, so we have a map

$$\mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \qquad f \mapsto f(\sigma)$$

Combining all of this, we define the **invariant map**

$$\mathrm{inv}_{L/K} : H^2(G, L^\times) \to \mathbb{Q}/\mathbb{Z}$$

as the composition

$$H^2(G, L^\times) \xrightarrow[\cong]{v_L} H^2(G, \mathbb{Z}) \xrightarrow[\cong]{\delta^{-1}} H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\cong]{} \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{f \mapsto f(\sigma)} \mathbb{Q}/\mathbb{Z}$$

Theorem 2.19 (Milne [2], Chapter 3, Theorem 1.7)**.** *Let K be a local field.*

1. *If K_n/K is finite unramified with $n = [K_n : K]$, then $\mathrm{inv}_{K_n/K}$ is an isomorphism*

$$\mathrm{inv}_{K_n/K} : H^2(\mathrm{Gal}(K_n/K), K_n^\times) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$$

2. *If $K \subset K_n \subset K_m$ is a tower of finite unramified extensions (this happens if $n|m$), then the following diagram commutes.*

$$\begin{array}{ccc} H^2(\mathrm{Gal}(K_n/K), K_n^\times) & \xrightarrow{\mathrm{inv}_{K_n/K}} & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \\ \Big\downarrow \mathrm{Inf} & & \Big\downarrow \\ H^2(\mathrm{Gal}(K_m/K), K_m^\times) & \xrightarrow{\mathrm{inv}_{K_m/K}} & \frac{1}{m}\mathbb{Z}/\mathbb{Z} \end{array}$$

That is to say, the maps $\mathrm{inv}_{L/K}$ give an isomorphism of directed systems between $H^2(\mathrm{Gal}(K_n/K), L^\times)$ for K_n/K finite with Inf *maps, and the system $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 1}$ with inclusion maps when $n|m$, inducing an isomorphism on the direct limit,*

$$\mathrm{inv}_K : H^2(\mathrm{Gal}(K^{\mathrm{un}}/K), K^{\mathrm{un}\,\times}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

Proof. (1) To show $\mathrm{inv}_{K_n/K}$ is injective, we just need to consider the kernel of

$$\mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{f \mapsto f(\sigma)} \mathbb{Q}/\mathbb{Z}$$

Recall that σ is a topological generator for G (the subgroup generated by σ) is dense, so if $f(\sigma) = 1$, then by continuity f is the trivial map. Thus $\mathrm{inv}_{K_n/K}$ is injective. (This works in the infinite case too, though we don't need this.) For K_n/K finite, we have $G = \mathbb{Z}/n\mathbb{Z}$, and homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ can only take the n-torsion element σ to n-torsion elements of $\mathbb{Q}/\mathbb{Z}$, so the image is the subgroup $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

(2) To see that $\mathrm{inv}_{L/K}$ commutes with Inf, it suffices to check that each map in the composition defining $\mathrm{inv}_{L/K}$ commutes with Inf. For the first three isomorphisms, this is "obvious" because Inf always commutes with maps in a long exact sequence on cohomology. For the final map, we just think about what Inf does in terms of cochains.

$$\mathrm{Inf} : \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(K_n/K), K_n^\times) \to \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Gal}(K_m/K), K_m^\times) \qquad f \mapsto (x \mapsto f(\overline{x}))$$

where $x \in \mathrm{Gal}(K_m/K)$, and $\overline{x} = x \bmod \mathrm{Gal}(K_m/K_n) \in \mathrm{Gal}(K_n/K) \cong \frac{\mathrm{Gal}(K_m/K)}{\mathrm{Gal}(K_m/K_n)}$. In terms of this description, we can see that the square commutes. Let $\sigma_m \in \mathrm{Gal}(K_m/K), \sigma_n \in \mathrm{Gal}(K_n/K)$ be the respective Frobenius elements (generators).

$$\mathrm{inv}_{K_m/K} \circ \mathrm{Inf}(f) = \mathrm{inv}_{K_m/K}\left(x \mapsto f(\overline{x})\right) = f(\sigma_m|_{K_n}) = f(\sigma_n) = \mathrm{inv}_{K_n/K)}(f)$$

This works out because σ_m restricts to σ_n on K_n, these are equal. ☐

Remark 2.20. This computes all of the relative Brauer groups of unramified extensions of K, including the maximal unramified extension.

$$\mathrm{Br}(K^{\mathrm{un}}/K) \cong \mathbb{Q}/\mathbb{Z}$$

2.4 Interpretation of multiplication by n

Ok, so we know that $\mathrm{Br}(K^{\mathrm{un}}/K) \cong \mathbb{Q}/\mathbb{Z}$ for a finite field K. Remember that if we have a finite extension L/K with $[L : K] = n$, we have the tensor homomorphism,

$$\epsilon : \mathrm{Br}(K^{\mathrm{un}}/K) \to \mathrm{Br}(L^{\mathrm{un}}/L) \qquad [A] \mapsto [A \otimes_K L]$$

which is now a homomorphism $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. The kernel of this is $\mathrm{Br}(L/K)$, which corresponds to the subgroup $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, so ϵ must correspond to the multiplication-by-n map on $\mathbb{Q}/\mathbb{Z}$.

We can also describe this correspondence more directly, which is the content of the next theorem. Hmm... or we could skip everything that follows this, because it follows more easily from what I just said here.

Remark 2.21. By work that Igor does in his notes [3], it turns out that $\mathrm{Br}(K^{\mathrm{un}}/K)$ is all of $\mathrm{Br}(K)$, so the Brauer group of a local field is $\mathbb{Q}/\mathbb{Z}$.

2.4.1 Unnecessary stuff for quick proof

Remark 2.22. For a local field K, the maximal unramified extension K^{un}/K is formed by adjoining all mth-roots of unity with m not divisible by the characteristic of the residue field k. To see this, recall the correspondence between finite unramified extensions of K and finite extensions of k.

$$\begin{array}{ccc} K^{\mathrm{un}} = \bigcup K_n & & \overline{\mathbb{F}}_p = \bigcup \mathbb{F}_{p^n} \\ \Big| & & \Big| \\ K_n & \leftrightsquigarrow & \mathbb{F}_{p^n} \\ \Big| & & \Big| \\ K & \leftrightsquigarrow & \mathbb{F}_p \end{array}$$

Recall that $\mathbb{F}_{p^n}$ is formed by adjoining all roots of $x^{p^n} - x$, which is to say, all $(p^n - 1)$th roots of unity. So $\overline{\mathbb{F}}_p$ is formed by adjoining all $p-1, p^2-1, p^3-1$ etc. roots of unity. So K^{un} is similarly formed by adjoinining various roots of unity. [2]

Remark 2.23. As a consequence of the previous remark, if L/K is a finite extension of local fields, then L^{un} is the compositum of L and K^{un}, since the maximal unramified extensions of each are formed by adjoining all roots of unity not dividing the characteristic of the residue fields.

Remark 2.24. Let L/K be a finite extension of local fields, let $\Gamma_K = \mathrm{Gal}(K^{\mathrm{un}}/K)$, $\Gamma_L = \mathrm{Gal}(L^{\mathrm{un}}/L)$. There is a restriction map

$$\mathrm{res} : \Gamma_L \to \Gamma_K \qquad \tau \mapsto \tau|_{K^{\mathrm{un}}}$$

This makes sense because of τ fixes L, it certainly fixes K. Note that because $L^{\mathrm{un}} = LK^{\mathrm{un}}$, it is injective, because if $\tau|_L = \mathrm{Id}$ and $\tau|_{K^{\mathrm{un}}} = \mathrm{Id}$, it is the identity on the compositum $LK^{\mathrm{un}} = L^{\mathrm{un}}$.

[2] I am mildly suspicious about this chain of reasoning.

Let σ_L, σ_K be the respective Frobenius elements of Γ_L, Γ_K, which means that they restrict to the Frobenius generator on any finite subextension. That is, $\sigma_K(x) = x^q$ on $k = \mathbb{F}_q$, and $\sigma_L(x) = x^{q^f}$ on $\ell = \mathbb{F}_{q^f}$, where $f = [\ell : k]$ is the residual field degree. Thus $\operatorname{res}(\sigma_L) = \sigma_K^f$.

Definition 2.25. Let L/K be a finite extension of local fields, and let Γ_K, Γ_L be as above. We define

$$\operatorname{Res} : H^2(\Gamma_K, K^{\mathrm{un}\,\times}) \to H^2(\Gamma_L, L^{\mathrm{un}\,\times}) \qquad [\phi] \mapsto [\iota \circ \phi \circ \operatorname{res}^2]$$

where $\iota : K^{\mathrm{un}\,\times} \hookrightarrow L^{\mathrm{un}\,\times}$ is the inclusion. This is essentially the same as the usual restriction map for group cohomology. Really what's going on is that this is the composition

$$H^2(\Gamma_K, K^{\mathrm{un}\,\times}) \xrightarrow{\widetilde{\operatorname{Res}}} H^2(\operatorname{Gal}(K^{\mathrm{un}}/L), K^{\mathrm{un}\,\times}) \to H^2(\Gamma_L, L^\times)$$

where $\widetilde{\operatorname{Res}}$ is the usual group cohomology restriction map $[\phi] \mapsto [\phi|_{\operatorname{Gal}(K^{\mathrm{un}}/L)}]$, and the second map is $[\phi] \mapsto [\iota \circ \phi \circ \operatorname{res}^2]$.

2.4.2 Main result interpreting multiplication by n, can skip long proof

Proposition 2.26 (Milne [2], Chapter 3, Proposition 1.8)**.** *Let L/K be a finite extension of local fields with $n = [L : K]$, and let Γ_K, Γ_L be as above. The following diagram commutes.*

$$\begin{array}{ccccc}
\operatorname{Br}(K^{\mathrm{un}}/K) & \xrightarrow[\cong]{\beta^{-1}_{K^{\mathrm{un}}/K}} & H^2(\Gamma_K, K^{\mathrm{un}\,\times}) & \xrightarrow[\cong]{\mathrm{inv}_K} & \mathbb{Q}/\mathbb{Z} \\
{\scriptstyle -\otimes_K L}\downarrow & & \downarrow{\scriptstyle \operatorname{Res}} & & \downarrow{\scriptstyle n} \\
\operatorname{Br}(L^{\mathrm{un}}/L) & \xrightarrow[\cong]{\beta^{-1}_{L^{\mathrm{un}}/L}} & H^2(\Gamma_L, L^{\mathrm{un}\,\times}) & \xrightarrow[\cong]{\mathrm{inv}_L} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Proof. (Quick proof.) We already know commutativity of the left square, and commutativity of the outer rectangle, so we're done. □

Proof. (Direct proof of commutativity of right square.) Let e, f be the ramification index and residual field degree for L/K. From the definition of $\mathrm{inv}_K, \mathrm{inv}_L$, to show commutativity of the second square, we need commutativity of the following three squares.

$$\begin{array}{ccccccc}
H^2(\Gamma_K, K^{\mathrm{un}\,\times}) & \xrightarrow{v_K} & H^2(\Gamma_K, \mathbb{Z}) & \xrightarrow{\delta^{-1}} & H^1(\Gamma_K, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{f \mapsto f(\sigma_K)} & \mathbb{Q}/\mathbb{Z} \\
\downarrow{\scriptstyle \operatorname{Res}} & & \downarrow{\scriptstyle e \operatorname{Res}} & & \downarrow{\scriptstyle e \operatorname{Res}} & & \downarrow{\scriptstyle ef = n} \\
H^2(\Gamma_L, L^{\mathrm{un}\,\times}) & \xrightarrow{v_L} & H^2(\Gamma_L, \mathbb{Z}) & \xrightarrow{\delta^{-1}} & H^1(\Gamma_L, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{f \mapsto f(\sigma_L)} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

Recall that $ef = n$ by the theory of local fields. By definition of e, we have the commutative square

$$\begin{array}{ccc}
K^{\mathrm{un}\,\times} & \xrightarrow{v_K} & \mathbb{Z} \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle e} \\
L^{\mathrm{un}\,\times} & \xrightarrow{v_L} & \mathbb{Z}
\end{array}$$

It follows that the leftmost square commutes. Here are some confusing symbols trying to explain why this follows. If they don't help you, ignore them.

$$v_L \operatorname{Res}[\phi] = [v_L \circ \iota \circ f \circ \operatorname{res}^2] = [e \circ v_K \circ \phi \circ \operatorname{res}^2] = e\, v_K \operatorname{Res}[\phi]$$

The second square, ignoring the e's, just says that regular Res commutes with boundary maps, which we know, so the middle square commutes. Again ignoring e's, the rightmost square is

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma_K, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\phi \mapsto \phi(\sigma_K)} & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle \phi \mapsto \phi \circ \mathrm{res}} \downarrow & & \downarrow {\scriptstyle f} \\ \mathrm{Hom}(\Gamma_L, \mathbb{Q}/\mathbb{Z}) & \xrightarrow[\phi \mapsto \phi(\sigma_L)]{} & \mathbb{Q}/\mathbb{Z} \end{array}$$

As we remarked earlier, $\mathrm{res}(\sigma_L) = \sigma_K^f$, so this square commutes. Potentially confusing symbol explanation included below.

$$\Big(\phi \mapsto \phi(\sigma_L)\Big) \circ \Big(\phi \mapsto \phi \circ \mathrm{res}\Big) = \Big(\phi \mapsto \phi\big(\mathrm{res}(\sigma_L)\big)\Big) = \Big(\phi \mapsto \big(\phi(\sigma_K^f)\big)\Big) = \Big(\phi \mapsto f \circ \phi(\sigma_K)\Big)$$

Then multiplying both downward arrows by e, we obtain the rightmost square. $\square$

References

[1] James S. Milne. Algebraic number theory (v3.07), 2017. Available at www.jmilne.org/math/.

[2] J.S. Milne. Class field theory (v4.02), 2013. Available at www.jmilne.org/math/.

[3] Igor Rapinchuk. The brauer group of a field. Available at https://drive.google.com/file/d/0B0CCc00SqXL4dTBIbU8xa0Vjb2c/edit.